7 Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Write the number of the question in this column

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Suppose that v_1, v_2, v_3, v_4 is basis of V.

Claim 1: $(v_1+v_2), (v_2+v_3), (v_3+v_4), v_4$ are linearly independent.

Suppose that there exist $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that

$$a_1(v_1+v_2)+a_2(v_2+v_3)+a_3(v_3+v_4)+a_4 v_4=0$$

$$a_1 v_1+(a_1+a_2)v_2+(a_2+a_3)v_3+(a_3+a_4)v_4=0$$

~~Thus~~ Since v_1, v_2, v_3, v_4 are linearly independent

$a_1=0$, $a_1+a_2=0$ $a_2+a_3=0$, $a_3+a_4=0$

Thus $a_1=a_2=a_3=a_4=0$. Thus $(v_1+v_2), (v_2+v_3), (v_3+v_4), v_4$ are linearly independent.

Claim 2: $\operatorname{Span}(\gamma_1+\gamma_2, \gamma_2+\gamma_3, \gamma_3+\gamma_4, \gamma_4) = V$

Let $v \in V$. Then there exist $a, b, c, d \in \mathbb{F}$ such that

$$v = a\gamma_1 + b\gamma_2 + c\gamma_3 + d\gamma_4$$

We rewrite

$$\gamma_3 = (\gamma_3 + \gamma_4) - \gamma_4$$

$$\gamma_2 = (\gamma_2 + \gamma_3) - (\gamma_3 + \gamma_4) + \gamma_4$$

$$\gamma_1 = (\gamma_1 + \gamma_2) - (\gamma_2 + \gamma_3) + (\gamma_3 + \gamma_4) - \gamma_4$$

Thus

$$\begin{aligned} v = {} & a((\gamma_1+\gamma_2) - (\gamma_2+\gamma_3) + (\gamma_3+\gamma_4) - \gamma_4) \\ & + b((\gamma_2+\gamma_3) - (\gamma_3+\gamma_4) + \gamma_4) + \\ & + c((\gamma_3+\gamma_4) - \gamma_4) + d\gamma_4 \\ = {} & a(\gamma_1+\gamma_2) + (-a+b)(\gamma_2+\gamma_3) \\ & + (a-b+c)(\gamma_3+\gamma_4) \\ & + (-a+b-c+d)\gamma_4 \end{aligned}$$

Thus $(\gamma_1+\gamma_2), (\gamma_2+\gamma_3), (\gamma_3+\gamma_4), \gamma_4$

spans V.

By claim 1 and 2 we get

$$(\gamma_1+\gamma_2), (\gamma_2+\gamma_3), (\gamma_3+\gamma_4), \gamma_4$$

basis for V

8 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Counter example:

Let $V = \mathbb{R}^4$ and $U := \{(x, y, z, 0) \in \mathbb{R}^4\}$

Let $v_1 := (1,0,0,0)$

$v_2 := (0,1,0,0)$

$v_3 := (0,0,1,1)$

$v_4 := (0,0,0,1)$

Thus $v_1, v_2 \in U$ and $v_3, v_4 \notin U$

But v_1, v_2 is not a basis of U.

Becaue $(0,0,1,0) \in U$ but v_1, v_2 does not span U

$(0,0,1,0) \neq a(1,0,0,0) + b(0,1,0,0)$

for all $a, b \in \mathbb{R}$.

9 Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $v_1, \dots, v_m$ is a basis of V if and only if $w_1, \dots, w_m$ is a basis of V.

Suppose that $v_1, \dots, v_m \in V$

Let $w_k = v_1 + \dots + v_k$ $\forall k \in \{1, 2, \dots, m\}$

forward direction

Suppose that $v_1, \dots, v_m$ is a basis of V.

Claim1: $w_1, w_2, \dots, w_m$ are linearly independent

Suppose that there exist $a_1, \dots, a_m \in \mathbb{F}$ such that

$$a_1 w_1 + \dots + a_m w_m = 0$$

$$a_1 v_1 + a_2(v_1 + v_2) + \dots + a_m(v_1 + \dots + v_m) = 0$$

$$(a_1 + \dots + a_m) v_1 + (a_2 + \dots + a_m) v_m + \dots + a_m v_m = 0$$

Since $v_1, \dots, v_m$ are linearly independent.

$$(a_1 + \dots + a_m) = 0, (a_2 + \dots + a_m) = 0, \dots, a_m = 0$$

This implies $a_1 = a_2 = \dots = a_m = 0$

Thus $w_1, \dots, w_m$ are linearly independent.

Claim2: $\operatorname{span}(w_1, \dots, w_m) = V$

Let $v \in V$. Since $\operatorname{span}(v_1, \dots, v_m) = V$, there exist $b_1, b_2, \dots, b_m \in \mathbb{F}$ such that

$$v = b_1 v_1 + \dots + b_m v_m$$

We rewrite
$v_1 = w_1$
$v_2 = w_2 - w_1$
$v_3 = w_3 - w_2$
$\vdots$
$v_m = w_m - w_{m-1}$

$$v = b_1 w_1 + b_2(w_2 - w_1) + \dots + b_m(w_m - w_{m-1})$$

$$= (b_1 - b_2) w_1 + (b_2 - b_3) w_2 + \dots + b_m w_m$$

Thus $\operatorname{span}(w_1, \dots, w_m) = V$. Therefore $w_1, w_2, \dots, w_m$ is a basis.

Backward direction.

Now suppose that $w_1, \dots, w_m$ be a basis for V.

Claim 3: $v_1, \dots, v_m$ are linearly independent.

Suppose that $c_1, \dots, c_m \in \mathbb{F}$ such that

$$c_1 v_1 + \dots + c_m v_m = 0$$

$$c_1 w_1 + c_2 (w_2 - w_1) + \dots + c_m (w_m - w_{m-1}) = 0$$

$$(c_1 - c_2) w_1 + \dots + c_m w_m = 0$$

Since $w_1, \dots, w_m$ are linearly independent,

$c_1 = c_2 = \dots = c_m = 0$. Thus $v_1, \dots, v_m$ are linearly independent.

Claim 4: ~~$\gamma_1, \dots, \gamma_m$~~ $\operatorname{span}(\gamma_1, \dots, \gamma_m) = V$

Let $w \in V$. Since $w_1, \dots, w_m$ is a basis of V there exist $d_1, \dots, d_m \in \mathbb{F}$ such that

$$w = d_1 w_1 + \dots + d_m w_m$$

$$w = d_1 v_1 + \dots + d_m (v_1 + \dots + v_m)$$

$$= (d_1 + \dots + d_m) v_1 + \dots + d_m v_m$$

Thus $\operatorname{span}(v_1, \dots, v_m) = V$

10 Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V.

Let V be a Vector space.
Let U, W are subspace of V.
Such that $V = U \oplus W$. Then $V = U + W$ and $U \cap W = \{0\}$.

Suppose that $u_1, \dots, u_m$ be a basis of U and $w_1, \dots, w_n$ be a basis of W

Claim: $u_1, \dots, u_m, w_1, \dots, w_n$ are linearly independent.

Suppose that $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that

$$a_1 u_1 + \dots + a_m u_m + b_1 w_1 + \dots + b_n w_n = 0$$

$$(a_1 u_1 + \dots + a_m u_m) = (-b_1) w_1 + (b_2) + \dots + (-b_n) w_n$$

Then

$$(a_1 u_1 + \dots + a_m u_m) = (-b_1) w_1 + \dots + (-b_n) w_n \in U \cap V = \{0\}$$

Thus

$a_1 u_1 + \dots + a_m u_m = 0$. Since $u_1, \dots, u_m$ are linearly independent $a_1 = a_2 = \dots = a_m = 0$

$(-b_1) w_1 + \dots + (-b_n) w_n = 0$. Since $w_1, \dots, w_n$ are linearly independent $b_1 = b_2 = \dots = b_n = 0$

Hence $u_1, \dots, u_m, w_1, \dots, w_n$ are linearly Independent

Claim 2: $\operatorname{span}(u_1, \ldots, u_m, w_1, \ldots, w_n) = V$

Let $v \in V$. Since $V = U \oplus W$, there exist $u \in U$ and $w \in W$ such that,

$$v = u + w$$

Since $u_1, \ldots, u_m$ spans U, then $u = c_1 u_1 + \cdots + c_m u_m$

Since $w_1, \ldots, w_n$ spans W, then $w = d_1 w_1 + \cdots + d_n w_n$

Thus $v = c_1 u_1 + \cdots + c_m u_m + d_1 w_1 + \cdots + d_n w_n$

Thus $u_1, u_2, \ldots, u_m, w_1, \ldots, w_n$ is basis for V.

11 Suppose V is a real vector space. Show that if $v_1, \ldots, v_n$ is a basis of V (as a real vector space), then $v_1, \ldots, v_n$ is also a basis of the complexification $V_{\mathbf{C}}$ (as a complex vector space).

See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbf{C}}$.

8 Suppose V is a real vector space.

- The *complexification* of V, denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all $u_1, v_1, u_2, v_2 \in V$.

- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Suppose that V is a real vector space.

If $v_1, \ldots, v_n$ is a basis of V.

~~$v_1, \ldots, v_n$~~

Claim: $v_1, \ldots, v_n$ are linearly independent in $V_{\mathbb{C}}$.

~~Le~~ Suppose that there exist

$a_1+b_1 i, a_2+b_2 i, \dots, a_n+b_n i \in \mathbb{C}$ such that

$(a_1+b_1 i)\gamma_1 + \cdots + (a_n+b_n i)\gamma_n = (0+0i)$

$a_1\gamma_1 + \cdots + a_n\gamma_n + i(b_1\gamma_1 + \cdots + b_n\gamma_n) = 0+0i$

Then $a_1\gamma_1 + \cdots + a_n\gamma_n = \mathbf{0}$

Since $\gamma_1, \dots, \gamma_n$ are linearly independent and $a_1, \dots, a_n \in \mathbb{R}$, then $a_1 = a_2 = \cdots = a_n = 0$

~~Since~~

Furthur $b_1\gamma_1 + \cdots + b_n\gamma_n = \mathbf{0}$

Since $\gamma_1, \dots, \gamma_n$ are linearly independent and $b_1, \dots, b_n \in \mathbb{R}$ ~~are linearly~~ then

$b_1 = \cdots = b_n = 0$.

Thus,

$a_1+b_1 i = a_2+b_2 i = \cdots = a_n+b_n i = 0+0i$

Thus $\gamma_1, \dots, \gamma_n$ are linearly independent

the
this

Claim 2: $\gamma_1, \ldots, \gamma_n$ spans $V_{\mathbb{C}}$

Let $p + iq \in V_{\mathbb{C}}$

~~p3~~ We know that $p, q \in V$

~~p~~ Since $\gamma_1, \ldots, \gamma_n$ is a basis for V. Then

$p = \cancel{a} c_1 \gamma_1 + \cdots + c_n \gamma_n$ for some $c_1, \ldots, c_n \in \mathbb{R}$.

$q = d_1 \gamma_1 + \cdots + d_n \gamma_n$ for some $d_1, \ldots, d_n \in \mathbb{R}$

$$p + iq = c_1 \gamma_1 + \cdots + c_n \gamma_n + d_1 \gamma_1 + \cdots + d_n \gamma_n$$
$$= (c_1 + d_1)\gamma_1 + \cdots + (c_n + d_n)\gamma_n$$

Thus $\gamma_1, \cancel{\ast}, \ldots, \gamma_n$ spans $V_{\mathbb{C}}$

Therefore $\gamma_1, \ldots, \gamma_n$ ~~h~~ is a bas for $V_{\mathbb{C}}$